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NECESSARY OPTIMALITY CONDITION FOR THE TIME OF FIRST ABSORPTION

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We examine a linear pursuit problem under conditions of local convexity [1]. We derive the necessary condition for the optimality of the time of first absorption at all points of the space (global optimality). General sufficient conditions for the optimality of the pursuit time have been given in [2, 3].

1. Let a linear pursuit problem in an n-dimensional Euclidean space R be described by:

a) a linear vector differential equation

$$dz/dt = Cz - u + v, \qquad u = u(t) \in P, \qquad v = v(t) \in Q \tag{1.1}$$

where C is a constant square matrix of order n, u and v are vector-valued functions, measurable for $t \geqslant 0$, called the controls of the players (the pursuer and pursued, respectively), $P \subset R$ and $Q \subset R$ are convex compacta;

b) a terminal set M representable in the form $M = M_0 + W_0$, where M_0 is a linear subspace of space R, W_0 is some compact convex set in a space L being the orthogonal complement of M_0 in R .

We shall assume that Conditions 1 – 3 of [4] have been fulfilled for problem (1.1); we retain the notation of [4] in the present paper. We denote the matrix e^{tC} by $\Phi(t)$, and a unit ball in L (its boundary is a sphere K) by S.

2. Under additional assumptions on sufficient smoothness of the function $\lambda(z, t)$ the necessary optimality conditions in [4] can be rewritten in differential form. Namely, let $c(t)$ be the function given in Lemma 2 in [4] and let D be the collection of pairs $(z, t) \in R \times (0, +\infty)$ such that

$$\lambda(z, t) < c(t) \tag{2.1}$$

(The continuity of both sides of inequality (2.1) guarantees the openness of set D.) Then the following known lemmas (see [2, 5]) hold.

Lemma 1. If $(z, t) \in D$, there exists a unique vector $\psi(z, t) \in K$ such that

$$\pi\Phi(t)\, z = -\lambda(z, t)\, \psi(z, t) + W(t, \psi(z, t)) \tag{2.2}$$

the function $\psi(z, t)$ is continuously differentiable on D, and $\psi(z, T(z)) \equiv \varphi(z)$ if $0 < T(z) < +\infty$.

We denote $G(z, t) = Cz - u(t, \psi(z, t)) + v(t, \psi(z, t))$, where $u(r, \varphi)$ and $v(r, \varphi)$ are the functions given by Condition 1 in [4].

Lemma 2. The function $\lambda(z, t)$ is continuously differentiable on D,

$$\partial\lambda(z, t)/\partial z = -\Phi^*(t)\, \psi(z, t)$$

$$\partial\lambda(z, t)/\partial t = -(\psi(z, t) \cdot \pi\Phi(t)\, G(z, t))$$

here $(\Phi^*(t)$ is the operator adjoint to operator $\Phi(t)$. In particular,

$$\frac{\partial\lambda(z, t)}{\partial t} = \left(\frac{\partial\lambda(z, t)}{\partial z} \cdot G(z, t)\right)$$

Finally, we are easily convinced by direct calculation that the function $\lambda(z, t)$ is twice continuously differentiable on D, and

$$\frac{\partial^2\lambda(z, t)}{\partial t^2} = \left(\frac{\partial^2\lambda(z, t)}{\partial z \partial t} \cdot G(z, t)\right) \tag{2.3}$$

Let us assume that the function $\partial^2\lambda(z, t)/\partial t^2$ is differentiable on D.

Theorem 1. Let $z_0 \in R \setminus M$, $T_0 = T(z_0) < +\infty$ be such that

$$\frac{\partial\lambda(z_0, T_0)}{\partial t} = \frac{\partial^2\lambda(z_0, T_0)}{\partial t^2} = 0 \tag{2.4}$$

Then, if the time $T_0 = T(z_0)$ is optimal,

$$H(z_0) = \frac{\partial^3\lambda(z_0, T_0)}{\partial t^3} - \left(\frac{\partial^3\lambda(z_0, T_0)}{\partial z \partial t^2} \cdot G(z_0, T_0)\right) \geqslant 0 \tag{2.5}$$

Proof. Let

$$\varphi_0 = \varphi(z_0)$$

$$z(s) = \Phi(s)\left[z_0 - \int_0^s \Phi(-r)\, [u(T_0 - r, \varphi_0) - v(T_0 - r, \varphi_0)]\, dr\right]$$

Then it is easily verified that

$$\lambda(z(s), T_0 - s) = 0, \quad \pi\Phi(T_0 - s)\, z(s) = W(T_0 - s, \varphi_0) \tag{2.6}$$

so that (see (2.2)) $(z(s),\ T_0 - s) \in D$; $\psi(z(s),\ T_0 - s) = \varphi_0$ and

$$G(z(s),\ T_0 - s) = Cz(s) - u(T_0 - s,\ \varphi_0) + v(T_0 - s,\ \varphi_0) = dz(s)/ds \tag{2.7}$$

Therefore, in correspondence with (2.3) and (2.7),

$$\frac{d}{ds}\left(\frac{\partial\lambda(z(s), T_0 - s)}{\partial t}\right) = \left(\frac{\partial^2\lambda(z(s), T_0 - s)}{\partial z\partial t} \cdot \frac{\partial z(s)}{ds}\right) - \frac{\partial^2\lambda(z(s), T_0 - s)}{\partial t^2} = 0$$

hence, by virtue of (2.4),

$$\frac{\partial\lambda(z(s), T_0 - s)}{\partial t} \equiv 0, \qquad s \in [0, T_0] \tag{2.8}$$

Since $T(z_0)$ is optimal, by virtue of Theorem 2 in [4]

$$\lambda(z(s),\ \tau) \leqslant 0, \qquad \tau \in [0,\ T_0 - s]$$

which together with (2.6), (2.8) yields

$$\frac{\partial^2\lambda(z(s), T_0 - s)}{\partial t^2} \leqslant 0 \tag{2.9}$$

Combining (2.4) and (2.9) we obtain (see (2.7))

$$0 \geqslant \lim_{s \to +0} \frac{1}{s}\left[\frac{\partial^2\lambda(z(s), T_0 - s)}{\partial t^2} - \frac{\partial^2\lambda(z_0, T_0)}{\partial t^2}\right] =$$
$$\left(\frac{\partial^3\lambda(z_0, T_0)}{\partial z\partial t^2} \cdot \frac{\partial z(s)}{\partial s}\bigg|_{s=0}\right) - \frac{\partial^3\lambda(z_0, T_0)}{\partial t^3} = -H(z_0)$$

Q.E.D.

Lemma 3. Let $\mu(z,\ t)$ be a thrice continuously differentiable function, positive on D and let $F(z,\ t) = \mu(z,\ t)\cdot\lambda(z,\ t)$. Then, if $z_0 \in R \setminus M$ satisfies (2.4), then

$$Y(z_0) = \frac{\partial^3 F(z_0, T_0)}{\partial t^3} - \left(\frac{\partial^3 F(z_0, T_0)}{\partial z\partial t^2} \cdot G(z_0, T_0)\right) \equiv \mu(z_0, T_0)\cdot H(z_0)$$

Proof. Direct differentiation yields

$$\frac{\partial^3 F}{\partial z\partial t^2} = \frac{\partial\mu}{\partial z}\frac{\partial\lambda}{\partial t^2} + 2\frac{\partial^2\mu}{\partial z\partial t}\frac{\partial\lambda}{\partial t} + \frac{\partial^3\mu}{\partial z\partial t^2}\lambda + \mu\frac{\partial^3\lambda}{\partial z\partial t^2} + 2\frac{\partial\mu}{\partial t}\frac{\partial^2\lambda}{\partial z\partial t} + \frac{\partial^2\mu}{\partial t^2}\frac{\partial\lambda}{\partial z}$$

Accoeding to (2.4) the first, second and third terms vanish at the point $(z_0,\ T_0) \in D$. So that at this point

$$\left(\frac{\partial^3 F}{\partial z\partial t^2}\cdot G\right) = \mu\left(\frac{\partial^3\lambda}{\partial z\partial t^2}\cdot G\right) + 2\frac{\partial\mu}{\partial t}\left(\frac{\partial^2\lambda}{\partial z\partial t}\cdot G\right) + \frac{\partial^2\mu}{\partial t^2}\left(\frac{\partial\lambda}{\partial z}\cdot G\right)$$

The second term equals zero by virtue of (2.3) and (2.4), while the third equals zero by virtue of Lemma 2 and of (2.4). Analogously, we convince ourselves that

$$\frac{\partial^3 F(z_0, T_0)}{\partial t^3} = \mu(z_0, T_0)\frac{\partial^3\lambda(z_0, T_0)}{\partial t^3}$$

This completes the proof of the lemma.

From Lemma 3 and the obvious equivalence conditions

$$\frac{\partial^i\lambda(z_0, T_0)}{\partial t^i} = 0, \quad i = 0, 1, 2, \qquad \frac{\partial^i F(z_0, T_0)}{\partial t^i} = 0, \quad i = 0, 1, 2$$

it follows that Theorem 1 remains in force if in its statement the function $\lambda(z,\ t)$ is everywhere replaced by $F(z, t)$.

3. The necessary optimality condition in [4] is tied to a given point z_0. Below we present the necessary condition for the global optimality of the time $T(z)$ of the upper layer, i.e. the optimality of $T(z)$ at each point of space R.

Definition. A point $z_0 \in R$ is called "singular" if there exist numbers $0 < T_1 < T_0 < +\infty$ such that $(T_1 = T(z_0))$

$$\lambda(z_0, t) < 0, \quad t \in [0, T_1) \cup (T_1, T_0); \qquad \lambda(z_0, T_0) = 0 \tag{3.1}$$

$$|\Phi^*(T_1)\varphi_1| \Phi^*(T_0)\varphi_0 \neq |\Phi^*(T_0)\varphi_0| \Phi^*(T_1)\varphi_1 \tag{3.2}$$

$$(\varphi_0 \cdot \Phi(T_0)[u(T_0, \varphi_0) - v(T_0, \varphi_0) - u(T_1, \varphi_1) + v(T_1, \varphi_1)]) < 0 \tag{3.3}$$

Here $\varphi_0 = \psi(z_0, T_0)$, $\varphi_1 = \psi(z_0, T_1)$.

Theorem 2. Let a singular point z_0 exist for problem (1.1). Then in R we also find a point z_* for which the time $T(z_*) < +\infty$ is nonoptimal.

Proof. We set

$$b = |\Phi^*(T_0)\varphi_0| \Phi^*(T_1)\varphi_1 - |\Phi^*(T_1)\varphi_1| \Phi^*(T_0)\varphi_0$$

then (see (3.2)) $(\varphi_1 \cdot \Phi(T_1) b) > 0$. By virtue of the continuity of the function $\psi(z, t)$ (see Lemma 1 and inequality (3.1)) there exists $\varepsilon \in (0, T_0 - T_1) \cap (0, T_1)$ such that

$$(\psi(z_0, t) \cdot \Phi(t) b) > 0, \qquad t \in E_1 = [T_1 - \varepsilon, T_1 + \varepsilon]$$

We take an arbitrary infinitesimal sequence $\delta_i > 0$, $i = 1, 2, \ldots$ and we set

$$\lambda_i = \min |\lambda(z_0, t)| > 0, \qquad t \in [0, T_0 - \delta_i] \setminus E_1$$

$$\gamma = \max |(\psi(z_0, t) \cdot \Phi(t) b)|, \qquad t \in [0, T_0]$$

$$z_i = z_0 + \alpha_i b, \qquad i = 1, 2, \ldots$$

where $\alpha_i = {}^1/_2 \lambda_i \gamma^{-1} \to 0$ as $t \to \infty$. Then for all $t \in [0, T_0 - \delta_i]$

$$\begin{aligned}\lambda(z_i, t) \leqslant (\psi(z_0, t) \cdot [W(t, \psi(z_0, t)) - \pi\Phi(t)(z_0 + \alpha_i b)]) = \\ \lambda(z_0, t) - \alpha_i(\psi(z_0, t) \cdot \Phi(t) b) < 0\end{aligned} \tag{3.4}$$

Since $(\varphi_0 \cdot \Phi(T_0) b) < 0$, by virtue of Lemma 1 the inequality

$$\beta_i = (\psi(z_i, T_0) \cdot \Phi(T_0) b) < 0$$

is fulfilled for all sufficiently large i. Therefore (see Lemma 1, [4])

$$\begin{aligned}\lambda(z_i, T_0) = (\psi(z_i, T_0) \cdot [W(T_0, \psi(z_i, T_0)) - \pi\Phi(T_0)(z_0 + \alpha_i b)]) = \\ (\psi(z_i, T_0) \cdot [W(T_0, \psi(z_i, T_0)) - W(T_0, \varphi_0)]) - \alpha_i\beta_i > 0\end{aligned} \tag{3.5}$$

Comparing inequalities (3.4) and (3.5) we obtain $T_i = T(z_i) \in [T_0 - \delta_i, T_0]$ for all sufficiently large i, so that

$$T_i \to T_0, \qquad i \to \infty \tag{3.6}$$

In accordance with Lemma 1

$$\psi_i = \varphi(z_i) = \psi(z_i, T_i) \to \psi(z_0, T_0) = \varphi_0, \quad i \to \infty \tag{3.7}$$

Let us show that the time $T(z_i)$ is nonoptimal for sufficiently large i (this also completes the proof of the theorem). We assume the contrary; let T_i be optimal for any $i = 1, 2, \ldots$. We propose that the pursuer, starting from the point z_i, chooses his own control $u_i(r)$, $0 \leqslant r \leqslant \varepsilon_i = \alpha_i^{1/2}$ as follows:

$$u_i(r) \equiv u(T_1 + \varepsilon_i - r, \psi(z_i, T_i + \varepsilon_i))$$

By virtue of the assumed optimality of time $T(z_i)$ the pursued can so choose his own control $v_i(r)$, $0 \leqslant r \leqslant \varepsilon_i$, that the inequality

$$\lambda(z_i(\varepsilon_i), T_i - \varepsilon_i) \leqslant 0, \qquad \lambda(z_i(\varepsilon_i), T_1) < 0 \tag{3.8}$$

is fulfilled for the motion

$$z_i(t) = \Phi(t)\Big(z_i - \int_0^t \Phi(-r)[u_i(r) - v_i(r)]\,dr\Big), \quad t \in [0, \varepsilon_i]$$

Otherwise we would have either $T(z_i(\varepsilon_i)) < T_i - \varepsilon_i$ or $T(z_i(\varepsilon_i)) \leqslant T_1$ and, consequently, $T(z_i(\varepsilon_i)) < T_i - \varepsilon_i$ for all sufficiently large i, whence, according to Theorem 1 from [4], would follow the nonoptimality of $T(z_i)$, i.e. a contradiction.

Let us show that system (3.8) is nevertheless contradicted for all sufficiently large i and that, consequently, the assumption that $T(z_i)$ is optimal is false. We set

$$\varphi_i = \psi(z_i(\varepsilon_i), T_1), \qquad \theta_i = \psi(z_i, T_1 + \varepsilon_i); \qquad \lambda^i = \lambda(z_i, T_1 + \varepsilon_i)$$

After manipulations (see Lemma 2 of [4] and Lemma 1) the second of inequalities (3.8) yields

$$0 > \lambda(z_i(\varepsilon_i), T_1) = (\varphi_i \cdot [W(T_1, \varphi_i) - W(T_1, \theta_i)]) +$$

$$\lambda^i \cdot (\varphi_i \cdot \theta_i) + \Big(\varphi_i \cdot \int_{T_1}^{T_1+\varepsilon_i} \Phi(s)[v(s, \theta_i) - v_i(T_1 + \varepsilon_i - s)]\,ds\Big) \geqslant \tag{3.9}$$

$$c_1(1 - (\varphi_i \cdot \theta_i)) + \lambda^i \cdot (\varphi_i \cdot \theta_i) + \Big(\varphi_i \cdot \int_{T_1}^{T_1+\varepsilon_i} \Phi(s)[v(s, \theta_i) - v_i(T_1 + \varepsilon_i - s)]\,ds\Big)$$

$$(c_1 = c(T_1) > 0)$$

Selecting, if necessary, a subsequence from $\{\varepsilon_i\}_{i=1}^{\infty}$, we can assume that

$$\lim_{i\to\infty} \frac{1}{\varepsilon_i} \int_0^{\varepsilon_i} v_i(r)\,dr = v_0 \in Q$$

(set Q is a convex compactum !).

Further, by virtue of formula (3.4), for all sufficiently large i

$$|\lambda^i| \leqslant |\lambda(z_0, T_1 + \varepsilon_i)| + 2\varepsilon_i^2\gamma \tag{3.10}$$

Since $\lambda(z_0, t)$ is a differentiable function of parameter t, taking the maximum value $\lambda(z_0, T_1) = 0$ at the point $t = T_1$,

$$\frac{\lambda(z_0, T_1 + \varepsilon_i)}{\varepsilon_i} \to \frac{\partial\lambda(z_0, T_1)}{\partial t} = 0, \quad i \to \infty$$

The latter relation together with (3.10) yields

$$|\lambda^i|\,\varepsilon_i^{-1} \to 0, \qquad i \to \infty \tag{3.11}$$

Finally, since $z_i \to z_0$; $z_i(\varepsilon_i) \to z_0$, by virtue of Lemma 1 we have $\varphi_i \to \psi(z_0, T_1) = \varphi_1$ and $\theta_i \to \varphi_1$. Consequently, (the function $v(r, \varphi)$ is uniformly continuous on $[T_1, T_0] \times K$; see Condition 1 in [4])

$$\max_{T_1 \leqslant s \leqslant T_1+\varepsilon_i} |v(s, \theta_i) - v(T_1, \varphi_1)| \to 0, \qquad i \to \infty \tag{3.12}$$

Dividing inequality (3.9) by $\varepsilon_i > 0$ and passing to the limit as $i \to \infty$, we obtain, using (3.11) and (3.12),

$$0 \geqslant c_1 \lim_{i \to \infty} \frac{1-(\varphi_i \cdot \theta_i)}{\varepsilon_i} + (\varphi_1 \cdot \Phi(T_1)[v(T_1, \varphi_1) - v_0])$$

Both terms on the right-hand side of the obtained inequality are nonnegative, therefore, according to Condition 1 in [4], $v_0 = v(T_1, \varphi_1)$.

After manipulations the first of inequalities (3.8) yields (here $\chi_i = \psi(z_i(\varepsilon_i), T_i - \varepsilon_i)$ and $c = \min c(t)$, $t \in [T_1, T_0]$; $c > 0$ (see Lemma 2 in [4])

$$0 \geqslant \lambda(z_i(\varepsilon_i), T_i - \varepsilon_i) = \Big(\chi_i \cdot \Big[W(T_i - \varepsilon_i, \chi_i) - W(T_i - \varepsilon_i, \psi_i) -$$
$$\int_{T_i-\varepsilon_i}^{T_i} \Phi(s)[u(s, \psi_i) - v(s, \psi_i)]\, ds + \int_0^{\varepsilon_i} \Phi(T_i - r)[u_i(r) - v_i(r)]\, dr\Big]\Big) \geqslant$$
$$c \cdot (1 - (\chi_i \cdot \psi_i)) - \Big(\chi_i \cdot \int_{T_i-\varepsilon_i}^{T_i} \Phi(s)[u(s, \psi_i) - v(s, \psi_i)]\, ds\Big) + \tag{3.13}$$
$$\int_0^{\varepsilon_i} (\chi_i \cdot \Phi(T_i - r)[u(T_1 + \varepsilon_i - r, \theta_i) - v_i(r)])\, dr$$

Dividing both sides of (3.13) by $\varepsilon_i > 0$ and passing to the limit, we obtain

$$0 \geqslant c \lim_{i \to \infty} \frac{1-(\chi_i \cdot \psi_i)}{\varepsilon_i} - (\varphi_0 \cdot \Phi(T_0)[u(T_0, \varphi_0) - v(T_0, \varphi_0)]) +$$
$$(\varphi_0 \cdot \Phi(T_0)[u(T_1, \varphi_1) - v(T_1, \varphi_1)])$$

Here we have used the uniform continuity of the functions $u(r, \varphi)$ and $v(r, \varphi)$ on $[T_1, T_0] \times K$, formula (3.7), and the relations $\chi_i \to \varphi_0$, $\theta_i \to \varphi_1$, $i \to \infty$. Since the first term is nonnegative, the inequality obtained contradicts (3.3). The theorem is proved.

Thus, the necessary condition for the global optimality of time $T(z)$ consists in the absence of singular points z in space R.

4. Let us additionally assume that $0 \in P$, $0 \in Q$ and that

$$\dim M_P = \dim M_Q = \dim L \tag{4.1}$$

where M_P and M_Q are subspaces of lowest dimension containing P and Q, respectively. In [3] it was shown that the condition

$$u(r, \varphi) \equiv u(\varphi), \quad v(r, \varphi) \equiv v(\varphi), \quad r > 0, \quad \varphi \in K \tag{4.2}$$

together with the condition of complete sweeping is sufficient for the optimality of time $T(z)$.

Theorem 3. In order for (4.2) to hold, it is necessary and sufficient that there exist continuous positive scalar functions $f(r)$ and $g(r)$, $r > 0$, and linear homeomorphisms

$A : M_P \to L$ and $B : M_Q \to L$ such that for all $r > 0$

$$\pi\,\Phi\,(r)\,u = f\,(r)\,Au, \qquad u \in P \tag{4.3}$$

$$\pi\,\Phi\,(r)\,v = g\,(r)\,Bv, \qquad v \in Q$$

We carry out the proof for the parameter u (it is analogous for v). Let (4.3) be fulfilled. By definition,

$$0 < (\varphi\cdot\Phi\,(r)\,[u\,(r,\varphi) - u]) = f\,(r)\,(\varphi\cdot A\,[u\,(r,\varphi) - u]), \quad u\,(r,\varphi) \neq u \in P$$

so that $u\,(r,\varphi)$ gives a strict maximum to the expression $(\varphi\cdot Au)$, $u \in P$, whence (4.2) follows.

Conversely, let (4.2) be fulfilled. According to Condition 1 of [4] the interior of $\pi\,\Phi\,(r)\,P$ is nonempty in L, therefore, it follows from (4.1) that for any $r > 0$ the mapping $\pi\,(r) = \pi\,\Phi\,(r) : M_P \to L$ is a linear "onto" homeomorphism and, hence, its adjoint mapping $\tau\,(r) = \pi^*\,(r) : L \to M_P$ also is a linear "onto" mapping [6]. We take arbitrary $\psi \in M_P$, $|\psi| = 1$ and $r > 0$. Let $\varphi\,(\psi, r) \in K$ be such that

$$\psi = \frac{\tau\,(r)\,\varphi\,(\psi, r)}{|\tau\,(r)\,\varphi\,(\psi, r)|}.$$

Then for all $u \in P$ we have

$$(\psi\cdot u) = \frac{(\varphi\,(\psi, r)\cdot\pi\Phi\,(r)\,u)}{|\tau\,(r)\,\varphi\,(\psi, r)|}$$

The maximum of the right-hand side of this equality is reached on the unique vector $u\,(\varphi\,(\psi; r))$ and, consequently, also the maximum of the left-hand side which, however, does not depend on r. Therefore, $u\,(\varphi\,(\psi, r_1)) = u\,(\varphi\,(\psi, r_2))$ is fulfilled for any $r_1, r_2 > 0$. Further, by virtue of the local convexity of the surface $\pi\,\Phi\,(r)\,u\,(K)$ we can find $c_1 > 0$ (see Lemma 1, [4]) such that

$$0 = (\varphi\,(\psi, r_1)\cdot\pi\Phi\,(r)\,[u\,(\varphi\,(\psi, r_1)) - u\,(\varphi\,(\psi, r_2))]) \geqslant c_1\,(1 - (\varphi\,(\psi, r_1)\cdot\varphi(\psi, r_2))) \geqslant 0$$

Thus, $\varphi\,(\psi, r_1) = \varphi\,(\psi, r_2) \equiv \varphi\,(\psi)$ for any $r_1,\ r_2 > 0$ and $\psi \in M_P$, $|\psi| = 1$. So that

$$\psi \equiv \frac{\tau\,(r)\,\varphi\,(\psi)}{|\tau\,(r)\,\varphi\,(\psi)|}, \qquad r > 0 \tag{4.4}$$

We fix $T > 0$. Let $\varphi \in K$ and $\psi = \tau\,(T)\,\varphi / |\tau\,(T)\,\varphi|$. Then in accordance to what was said above, $\varphi = \varphi\,(\psi)$. Therefore, by virtue of (4.4),

$$\tau\,(r)\,\varphi \equiv f\,(r,\varphi)\,\tau\,(T)\,\varphi, \quad f\,(r,\varphi) = |\tau\,(r)\,\varphi| \,/\, |\tau\,(T)\,\varphi|$$

for any $r > 0$. Let us show that $f\,(r,\varphi)$ does not depend on φ. Indeed, let φ_1 and φ_2 be linearly independent vectors from K. Then the vectors $\tau\,(T)\,\varphi_1$ and $\tau\,(T)\,\varphi_2$ are linearly independent ($\tau\,(T)$ is a homeomorphism!). Therefore, from the trivial relation

$$f\left(r, \frac{\varphi_1+\varphi_2}{|\varphi_1+\varphi_2|}\right)\frac{\tau\,(T)\,\varphi_1 + \tau\,(T)\,\varphi_2}{|\varphi_1+\varphi_2|} = \tau\,(r)\,\frac{\varphi_1+\varphi_2}{|\varphi_1+\varphi_2|} = \frac{f\,(r,\varphi_1)\,\tau\,(T)\,\varphi_1 + f\,(r,\varphi_2)\,\tau\,(T)\,\varphi_2}{|\varphi_1+\varphi_2|}$$

we obtain

$$f\,(r,\varphi_1) = f\left(r, \frac{\varphi_1+\varphi_2}{|\varphi_1+\varphi_2|}\right) = f\,(r,\varphi_2)$$

for any $\varphi_1, \varphi_2 \in K$, $r > 0$. Thus, there exists a function $f(r) > 0$ such that

$$\tau(r)\,\varphi \equiv f(r)\,\tau(T)\,\varphi; \qquad r > 0, \qquad \varphi \in K \tag{4.5}$$

Multiplying (4.5) scalarly by $u \in P$ and using the arbitrariness of φ, we obtain $\pi\,\Phi(r)\,u \equiv f(r)\,\pi\,\Phi(T)\,u$ for all $u \in P$, $r > 0$. Hence, in particular, follows the continuity of $f(r)$. The theorem is proved.

5. Let

$$0 < T_0 = T(z_0) < +\infty, \qquad \varphi_0 = \varphi(z_0)$$

$$I(t, \tau) = \lambda\left(\Phi(t)\left[z_0 - \int_0^t \Phi(-r)\,[u(T_0 - r, \varphi_0) - v(T_0 - r, \varphi_0)]\,dr\right], \tau\right).$$

It was shown in [4] that the inequality

$$\mathrm{I}(t, \tau) \leqslant 0; \quad t \geqslant 0, \quad \tau \geqslant 0, \quad t + \tau \leqslant T_0 \tag{5.1}$$

must necessarily hold for the optimality of time $T(z_0)$. Let us consider one important special case for which the verification of inequality (5.1) is considerably simplified. We assume that the hypotheses of Theorem 3 are fulfilled for problem (1.1) and

$$AP = BQ = S \tag{5.2}$$

and also that

$$W_0 = lS, \qquad l \geqslant 0 \tag{5.3}$$

Then $\pi\Phi(r)\,u(\varphi) = f(r)\,\varphi$; $\pi\Phi(r)\,v(\varphi) = g(r)\,\varphi$; $W(t, \varphi) = h(t)\,\varphi$, where

$$h(t) = l + \int_0^t [f(r) - g(r)]\,dr > 0, \qquad t > 0$$

(see Condition 3, [4]); $\lambda(z, t) = h(t) - |\,\pi\Phi(t)\,z\,|$, $T_0 = T(z_0)$ is the smallest nonnegative root of the equation

$$F(z_0, t) = h^2(t) - |\pi\Phi(t)\,z_0|^2 = 0 \tag{5.4}$$

Here and subsequently we have used the function $F(z, t)$, more suitable than $\lambda(z, t)$, having the same roots. Finally, $\varphi(z_0) = \pi\Phi(T_0)\,z_0/h(T_0)$.

Theorem 4. If relations (5.2), (5.3) have been fulfilled for problem (1.1), then inequality (5.1) holds if and only if

$$I(t, T_i) \leqslant 0, \qquad 0 \leqslant t \leqslant T_0 - T_i, \quad i = 1, 2, \ldots, m \tag{5.5}$$

where $0 \leqslant T_1 \leqslant T_2 \leqslant \ldots \leqslant T_m$ are all points of local maximum of the function $h(t)$ considered on the interval $[0, T_0]$ (in particular, these points may turn out to be the endpoints of the interval).

Proof. The necessity of formula (5.5) is obvious. Let us prove the sufficiency. To the contrary, suppose that inequalities (5.5) have been fulfilled but that there exist $t_0 > 0$, $\tau_0 > 0$, $t_0 + \tau_0 < T_0$ such that

$$\lambda\left(\Phi(t_0)\left[z_0 - \int_0^{t_0} \Phi(-r)\,[u(\varphi_0) - v(\varphi_0)]\,dr\right], \tau_0\right) > 0$$

or equivalently (see (5.4)),

$$h^2(\tau_0) - |h(\tau_0)\varphi_0 - \Psi|^2 > 0, \qquad \Psi = h(\tau_0 + t_0)\varphi_0 - \pi\Phi(\tau_0 + t_0)z_0$$

Hence

$$-|\Psi|^2 + 2h(\tau_0)\alpha > 0, \qquad \alpha = (\varphi_0 \cdot \Psi) \tag{5.6}$$

and, in particular,

$$\alpha > 0 \tag{5.7}$$

From the definition of $T(z_0)$ and from the inequality $\tau_0 + t_0 < T_0$ we have (see (5.4))

$$h^2(\tau_0 + t_0)^2 - |\pi\Phi(\tau_0 + t_0)z_0 - (h(\tau_0 + t_0) - h(\tau_0 + \tau_0))\varphi_0|^2 < 0$$

or equivalently

$$-|\Psi|^2 + 2h(\tau_0 + t_0)\alpha < 0 \tag{5.8}$$

Subtracting inequality (5.8) from (5.6), we obtain

$$2\alpha(h(\tau_0) - h(\tau_0 + t_0)) > 0$$

whence, by virtue of (5.7), $h(\tau_0) > h(\tau_0 + t_0)$.

Let $T^* = T_{i0}$ be a point of local maximum of the function $h(t)$ on the interval $[0, T_0]$, such that $T^* < \tau_0 + t_0$ and $h(T^*) \geqslant h(\tau_0)$. Then (see (5.7))

$$2\alpha(h(T^*) - h(\tau_0)) \geqslant 0$$

Adding this inequality to (5.6) we have

$$h^2(T^*) - |\pi\Phi(\tau_0 + t_0)z_0 - (h(\tau_0 + t_0) - h(T^*))\varphi_0|^2 > 0$$

This is equivalent to the inequality $I(t^*, T_{i_0}) > 0$, where $t^* = \tau_0 + t_0 - T^* > 0$, which contradicts (5.5). The theorem is proved.

By a verbatim repetition of the arguments presented above it is easy to see that the assertion of Theorem 4 remains in force if inequalities (5.1) and (5.5) are replaced, respectively, by

$$I(t,\tau) < 0, \quad t \geqslant 0, \quad \tau \geqslant 0, \quad t + \tau < T_0 = T(z_0) \tag{5.9}$$

$$I(t, T_i) < 0, \quad t \geqslant 0, \quad t < T_0 - T_i, \quad i = 1, \ldots, m$$

6. It turns out that condition (5.9), even in combination with (5.2), (5.3), can prove to be insufficient for the optimality of time $T(z_0)$ if the condition of complete sweeping (see [3]) is not fulfilled on $[0, T_0]$.

Let us illustrate what we have said by an example. In problem (1.1) (for convenience we write column-vectors in rows) let $z_1, z_2, z_3, z_4, z_5, \psi, \theta$ be ν-dimensional vectors $(\nu \geqslant 4)$

$$z = (z_1, z_2, z_3, z_4, z_5), \quad |\psi| \leqslant 1, \quad |\theta| \leqslant 1, \quad \lambda = 10^9$$

$$u = (100\psi + \lambda\psi, 0, 840\psi, 0, 960\psi), \quad v = (\lambda\theta, 340\theta, 0, 1320\theta, 0)$$

$$C = \begin{Vmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{Vmatrix}, \qquad M = M_0 = \{z:\ z_1 = 0\}$$

Then, as is easily verified,

$$\pi\Phi(t)\,z = z_1 + tz_2 + \frac{t^2}{2}z_3 + \frac{t^3}{6}z_4 + \frac{t^4}{24}z_5$$

$$\pi\Phi(r)\,u = f(r)\,\psi, \qquad f(r) = 100 + \lambda + 420r^2 + 40r^4$$

$$\pi\Phi(r)\,v = g(r)\,\theta, \qquad g(r) = \lambda + 340r + 220r^3$$

$$h(t) = 23\,[1 + ({}^8/_{23}t - 1)(t-1)^4] > 0$$

So that conditions (5.2), (5.3) are fulfilled.

Let $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ be ν-dimensional unit vectors such that $(\varphi_0\cdot\varphi_1) = 21/23$ and $(\varphi_i\cdot\varphi_j) = 0$ for $i \neq j$ in the remaining cases. Let $\delta > 0$, $\mu = 2\sqrt{386}$. We consider the initial state $z_0 = (z_{10}, z_{20}, z_{30}, z_{40}, z_{50})$ where

$$z_{10} = 16\varphi_0 + \delta^2\varphi_3$$

$$z_{20} = -64\varphi_0 + 92\varphi_1 + \mu\varphi_2 - {}^1/_2\,\delta^2\varphi_3$$

$$z_{30} = 192\varphi_0 - 276\varphi_1 - 5\mu\varphi_2$$

$$z_{40} = -z_{50} = -384\varphi_0 + 552\varphi_1 + 12\mu\varphi_2$$

Then

$$\pi\Phi(t)\,z_0 = 16(1-t)^4\varphi_0 + 23\,[1-(1-t)^4]\,\varphi_1 + {}^1/_2\,\mu t(2-t)(1-t)^2\varphi_2 + {}^1/_2\,\delta^2(2-t)\,\varphi_3$$

Direct calculation yields

$$F(z_0, t) = h^2(t) - |\pi\Phi(t)\,z_0|^2 = -(2-t)^2\,[2(1-t)^4\,P_4(t) + {}^1/_4\delta^4]$$

$$P_4(t) = 32 + 240t - 79t^2 + 184t^3 - 32t^4 > 0 \text{ on } [0, 2]$$

So that $T_0 = T(z_0) = 2$, $\varphi(z_0) = \varphi_0$. Further, $T^* = 1$ is the single point of local maximum of function $h(t)$ on $[0, 2]$. Therefore, by virtue of Theorem 4, condition (5.9) also is fulfilled, because

$$F\left(\Phi(t)\left[z_0 - \int_0^t \Phi(-r)\,[u(\varphi_0) - v(\varphi_0)]\,dr\right],\ T^*\right) = -(1-t)^2\left[t^4Q_4(t) + \frac{\delta^4}{4}\right]$$

$$t \in (0,\ T_0 - T^*)$$

Here $Q_4(t) = 64t^4 - 32t^3 + 446t^2 + 924t + 630 > 0$ on $[0, 1]$.

However, the time $T_0 = T(z_0)$ is nonoptimal for sufficiently small δ (for examole, for $\delta = 10^{-8}$). The proof of this fact is easily carried out by contradiction by a verbatim repetition of the arguments in Sect. 3, by assuming that $T(z_0)$ is optimal and by proposing that on the interval $[0, \delta]$ the pursuer sets his own control equal to

$$\psi(r) \equiv \varphi^* = \frac{\pi\Phi(T^* + \delta)\,z_0}{|\pi\Phi(T^* + \delta)\,z_0|}$$

In other words, $u(r) \equiv u(\varphi^*)$, $0 \leqslant r \leqslant \delta$. The possibility for such a path to the proof is connected with the fact that the point z_0 is singular when $\delta = 0$.

7. For linear pursuit problems studied in Sect. 5, i.e. satisfying conditions (5.2), (5.3), what we said in Sect. 2 takes the following form:

$$D = \{(z, t) : |\pi\Phi(t)\,z| > 0\}, \qquad \psi(z, t) = \pi\Phi(t)\,z/\,|\pi\Phi(t)\,z|$$

$$\mu(z, t) = h(t) + |\pi\Phi(t)\,z|$$

so that the function $F(z, t)$ of Lemma 3 is given by the left-hand side of formula (5.4). By differentiating with respect to r from formulas (4.3), (5.2), (5.3) we obtain that

$$\begin{aligned} \pi\Phi(r)\,[u(\varphi) - v(\varphi)] &= h'(r)\,\varphi \\ \pi\Phi(r)\,C\,[u(\varphi) - v(\varphi)] &= h''(r)\,\varphi \\ \pi\Phi(r)\,C^2[u(\varphi) - v(\varphi)] &= h'''(r)\,\varphi \end{aligned} \tag{7.1}$$

Here we have used the relations

$$\Phi'(r) = \Phi(r)\,C, \qquad \Phi''(r) = \Phi(r)\,C^2, \qquad h'(r) = f(r) - g(r)$$

Condition (2.4) is rewritten as

$$\frac{\partial F(z_0, T_0)}{\partial t} = 2h(T_0)\,[h'(T_0) - (\varphi_0 \cdot \Psi_1)] = 0, \quad \Psi_1 = \pi\Phi(T_0)\,Cz_0 \tag{7.2}$$

$$\frac{\partial^2 F(z_0, T_0)}{\partial t^2} = 2\{h(T_0)\,h''(T_0) + [h'(T_0)]^2 - h(T_0)(\varphi_0 \cdot \Psi_2) - |\Psi_1|^2\} = 0$$

$$\Psi_2 = \pi\Phi(T_0)\,C^2 z_0$$

The first of these equalities yields ($|\varphi_0| = 1$)

$$|\pi\Phi(T_0)\,C\,z_0| \geqslant h'(T_0) \tag{7.3}$$

The function $F(z, t)$ is infinitely differentiable on D. By directly computing the left-hand side of (2.5) we obtain (see the definition of $G(z, t)$ and the formulas (7.1))

$$\begin{gathered} {}^1/_2 Y(z_0) = h(T_0)\,h'''(T_0) + 3h'(T_0)h''(T_0) - h(T_0)\,(\varphi_0 \cdot \pi\Phi(T_0)C^3 z_0 - \\ 3(\Psi_1 \cdot \Psi_2) + \{(\pi\Phi(T_0)\,G(z_0, T_0) \cdot \Psi_2) + h(T_0)(\varphi_0 \cdot \pi\Phi(T_0)\;C^2 G(z_0, T_0)) + \\ 2(\Psi_1 \cdot \pi\Phi(T_0)\,CG(z_0, T_0))\} = \\ 3h'(T_0)\,h''(T_0) - h'(T_0)\,(\varphi_0 \cdot \Psi_2) - 2h''(T_0)\,(\varphi_0 \cdot \Psi_1) \end{gathered}$$

Applying (7.2) we finally obtain

$$Y(z_0) = \frac{2h'(T_0)}{h(T_0)}\{|\pi\Phi(T_0)\,Cz_0|^2 - |h'(T_0)|^2\}$$

Therefore, if $h'(T_0) \geqslant 0$, then by virtue of (7.3) the necessary condition (2.5) is fulfilled. If, however, $h'(T_0) < 0$ and $|\pi\Phi(T_0)\,C\,z_0| > |h'(T_0)|$, then the time $T_0 = T(z_0)$ is nonoptimal.

Lemma 4. Suppose that conditions (5.2), (5.3) have been fulfilled for the pursuit problem (1.1). Let relation (3.1) be fulfilled for a point $z_0 \in R$, and $h'(T_0) < 0$. Then, in order that point z_0 be singular it is necessary and sufficient to fulfill the non-equality $\varphi_1 \neq \varphi_0$.

Proof. If $\varphi_1 = \varphi_0$, condition (3.3) is not fulfilled and, consequently, point z_0 is not singular. If, however, $\varphi_1 \neq \varphi_0$, then by virtue of (5.2), (5.3)

$$(\varphi_0 \cdot \Phi(T_0)\,\{u(T_0, \varphi_0) - v(T_0, \varphi_0) - u(T_1, \varphi_1) + v(T_1, \varphi_1)\}) =$$

$$h'(T_0)\,(1 - (\varphi_0\varphi_1)) < 0$$

so that inequality (3.3) is fulfilled.

Let us show that relation (3.2) also is valid. Let

$$\alpha\Phi^* (T_0)\, \varphi_0 = \beta\Phi^* (T_1)\, \varphi_1 \tag{7.4}$$

Here $\alpha = | \Phi^* (T_1)\, \varphi_1 |$, $\beta = | \Phi^* (T_0)\, \varphi_0 |$. Then multiplying (7.4) scalarly by $u\, (\varphi_0)$ and $u\, (\varphi_1)$, we obtain

$$\alpha f\, (T_0) = \beta f\, (T_1)\, (\varphi_0 \cdot \varphi_1), \qquad \alpha f\, (T_0)\, (\varphi_0 \cdot \varphi_1) = \beta f\, (T_1)$$

Hence $(\varphi_0 \cdot \varphi_1) = 1$ and so $\varphi_1 = \varphi_0$. A contradiction.

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IMPULSE TRACKING OF A POINT WITH BOUNDED THRUST

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We examine the game problem [1 – 3] of the contact of two material points with unit masses moving in a three-dimensional space under the action of only the controlling forces F_1 and F_2 arbitrary in direction. It is assumed that force F_1 is bounded in momentum, while F_2, in absolute value. In parallel we consider two problems on the minimax time up to "hard" (with respect to the coordinates) and up to "soft" (with respect to the coordinates and velocities) contact. In both problems the whole space of positions is divided into two regions. The optimal controls of the first (the minimizing) player (point) and of the second (the maximizing) player (point) are formed in the first region and the minimax time computed as well. The second player's control permitting him to evade contact under any action of the first player is formed in the second region. A comparison is made with a previously-considered case [4] in which both points can move along certain fixed straight lines.